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MULTIPLY PERFECT NUMBERS OF FOUR DIFFERENT PRIMES*

By R. D. Carmichael

A multiply perfect number, according to the definition given by Lehmer† is one which is exactly divisible into the sum of its divisors including itself, and the quotient is called the multiplicity. It is easily shown that if the number is of the form

$$N = p_1^{a_1} p_2^{a_2} p_3^{a_3} p_4^{a_4}$$

with only four different prime factors, then the multiplicity m cannot be greater than 4.‡ The numbers of this form with multiplicity equal to 4 have already been discussed,§ and it was found that there is only one, namely $2^5 \cdot 3^3 \cdot 5 \cdot 7 = 30240$. There remain to be investigated therefore only those for which the multiplicity is 2 or 3, since the case $m = 1$ is excluded by the definition.

In the following pages it will be shown that the only even multiply perfect numbers of multiplicity 2 are of the known form $2^{a_1}(2^{a_1+1} - 1)$ where $2^{a_1+1} - 1$ is a prime, so that there are no even multiply perfect numbers of multiplicity 2 having four different primes. Furthermore it turns out that there is but one number of multiplicity 3, namely $2^9 \cdot 3 \cdot 11 \cdot 31$, and that there are no odd multiply perfect numbers of four different primes.

1. Even multiply perfect numbers of multiplicity 2. From the definition of a multiply perfect number it follows at once that

$$(1) \qquad m = \frac{p_1^{a_1} + p_1^{a_1-1} + \cdots + 1}{p_1^{a_1}} \cdots \frac{p_4^{a_4} + p_4^{a_4-1} + \cdots + 1}{p_4^{a_4}}$$

$$= \frac{p_1^{a_1+1} - 1}{p_1^{a_1}(p_1 - 1)} \cdots \frac{p_4^{a_4+1} - 1}{p_4^{a_4}(p_4 - 1)}.$$

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* Presented to the American Mathematical Society, October 27, 1906.

† Annals of Mathematics, ser. 2, vol. 2, p. 103.

‡ See Lehmer, Annals of Mathematics, ser. 2, vol. 2, p. 1; or this can be seen from the inequality (2) by substituting 2, 3, 5, 7 for p_1, p_2, p_3, p_4 respectively.

§ Carmichael, *American Mathematical Monthly*, vol. 13, p. 88; 1906.

Consequently

$$m < \frac{p_1}{p_1 - 1} \cdots \frac{p_4}{p_4 - 1}. \tag{2}$$

We shall in the present section consider n factors instead of four. It is readily seen what (1) and (2) now become. In equation (1), putting $m = 2$, $p_1 = 2$ and reducing, we have

$$\frac{2^{\alpha_1+1}}{2^{\alpha_1+1} - 1} = \frac{p_2^{\alpha_2} + \cdots + p_2 + 1}{p_2^{\alpha_2}} \cdots \frac{p_n^{\alpha_n} + \cdots + p_n + 1}{p_n^{\alpha_n}}. \tag{3}$$

Hence

$$1 + \frac{1}{2^{\alpha_1+1} - 1} \gtreqqless \left(1 + \frac{1}{p_2}\right)\left(1 + \frac{1}{p_3}\right) \cdots \left(1 + \frac{1}{p_n}\right). \tag{4}$$

The only factors which $2^{\alpha_1+1} - 1$ can have are among the primes p_k. But if p_k is a factor of $2^{\alpha_1+1} - 1$,

$$1 + \frac{1}{p_k} > 1 + \frac{1}{2^{\alpha_1+1} - 1};$$

and hence (4) is not satisfied. Therefore $2^{\alpha_1+1} - 1$ must equal one of the p's. Hence one factor of the right member of (4) is equal to its left member. Therefore the right member cannot contain a second factor. Then $n = 2$ and $p_2 = 2^{\alpha_1+1} - 1$. Also, equation (3) can be satisfied only when $\alpha_2 = 1$. Hence,

All multiply perfect even numbers of multiplicity 2 are of the form $2^{\alpha_1}(2^{\alpha_1+1} - 1)$ where $2^{\alpha_1+1} - 1$ is prime; and conversely all such numbers are multiply perfect numbers of multiplicity 2.

2. Even multiply perfect numbers of the form $2^{\alpha_1} 3^{\alpha_2} p_3^{\alpha_3} p_4^{\alpha_4}$ and multiplicity 3. If $m = 3$ and $p_1 = 2$, it is easy to show from (2) that $p_2 = 3$ or 5. Hence we have as a first result that *if a number of the type $2^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} p_4^{\alpha_4}$ is multiply perfect the value of p_2 is either 3 or 5.* In this section we shall consider only the case when $p_2 = 3$. From equation (1) we have

$$3 = \frac{2^{\alpha_1+1} - 1}{2^{\alpha_1}} \cdot \frac{3^{\alpha_2+1} - 1}{2 \cdot 3^{\alpha_2}} \cdot \frac{p_3^{\alpha_3+1} - 1}{p_3^{\alpha_3}(p_3 - 1)} \cdot \frac{p_4^{\alpha_4+1} - 1}{p_4^{\alpha_4}(p_4 - 1)}. \tag{5}$$

For definiteness let us say that $p_4 > p_3$. We shall first determine some limitations which are to be placed on the exponents $a_3 + 1$ and $a_4 + 1$. The following lemma will be of frequent use:

Lemma. *If x is a positive integer > 1, $x^t - 1$ has a prime factor not dividing $x^u - 1$ $(u < t)$, except in the cases $t = 2$, $x = 2^r - 1$, $v \geqq 2$; $t = 6$, $x = 2$.** *Such prime factors of $x^t - 1$ are of the form $st + 1$, and evidently if t is odd and > 1 they are of the form $2st + 1$.*† Such a prime $st + 1$ we shall call a characteristic factor.

Suppose that in equation (5) the exponent $a_4 + 1$ contains an odd prime factor > 1, say r, and let $a_4 + 1 = ry$, where y is to be determined. It may be shown by the lemma that in every case for which $y \neq 1$, $p_4^{ry} - 1$ contains more than one prime factor $\geqq 7$; for $p_4^r - 1$ and $p_4^{ry} - 1$ contain at least one characteristic factor each. But since in this case equation (5) can not be satisfied it follows that $y = 1$. That is, if $a_4 + 1$ contains an odd factor it must equal an odd prime. In the same way we may show that if $a_3 + 1$ contains an odd factor it must also be an odd prime.

It is evident at once from the theory of congruences that every prime divisor of $(p_4^{a_4+1} - 1) / (p_4 - 1)$, $a_4 + 1$ an odd prime, is a characteristic factor; and therefore it readily follows from equation (5) that

$$p_4^{a_4+1} - 1 = (p_4 - 1)\, p_3^{a}, \tag{6}$$

where a is some integer $> a_4$, p_4 being $> p_3$. It is easy to show that when $a_3 + 1$ and $a_4 + 1$ each is an odd prime, neither of the expressions $(p_3^{a_3+1} - 1) / (p_3 - 1)$, $(p_4^{a_4+1} - 1) / (p_4 - 1)$ in (5) can contain either 2 or 3; for every prime factor of each is a characteristic factor $\geqq 7$. Then from (5) we have

$$1 = \left(1 - \frac{1}{2^{a_1+1}}\right)\left(1 - \frac{1}{3^{a_2+1}}\right) p_3^{\mu}\, p_4^{\nu}. \tag{7}$$

It is readily shown that this equation is not satisfied if either μ or ν is greater than zero or if both equal zero. Therefore it follows that one of the exponents $a_3 + 1$, $a_4 + 1$ can contain no prime factor except 2.

Let us take $a_3 + 1 = 2^{\mu}$ and let us still regard $a_4 + 1$ as equal to some odd prime integer. We should yet have as before equation (6) above; and since

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* Dickson, "On the cyclotomic function," *American Mathematical Monthly*, vol. 12, p. 89, 1905.

† *Encyklopädie der Mathematischen Wissenschaften*, vol. 1, p. 577.

$a > a_4$ we have $a \geqq 3$. But $a_3 \geqq a$ and therefore $a_3 + 1 \geqq 4$ and it follows that $\mu \geqq 2$. Moreover it is easy to show that if $a_4 + 1 \geqq 5$ then $\mu \geqq 3$. Now in every case for which $\mu \geqq 3$, the numerator $p_3^{a_3+1} - 1$ in equation (5) contains $p_3^8 - 1$. By the lemma, $p_3^4 - 1$, $p_3^8 - 1$ each contains a characteristic factor $\geqq 5$ and prime to p_3; but at most two primes $\geqq 5$ may be introduced and this case is therefore excluded, p_3 being $\geqq 5$. Then consider the case $\mu = 2$ and $a_4 + 1 = 3$. Now $a_3 = 3$, $a_4 = 2$, and $a > a_4$ and $\leqq a_3$; and therefore $a = 3$. Then from (6) we have

$$p_4^2 + p_4 + 1 = p_3^3.$$

Moreover every prime factor of $p_3^2 + p_3 + 1$ is a characteristic factor $\geqq 7$; and therefore p_4 is the only such factor. Hence

$$p_3^2 + p_3 + 1 = p_4 \text{ or } p_4^2,$$

neither case of which satisfies the next preceding equation. This case is therefore excluded. Therefore $a_3 + 1$ can not be a power of 2 when $a_4 + 1$ is not a power of 2. Combining this result with that of the preceding paragraph we are able to announce that $a_4 + 1 =$ some power of 2. In the same way as the condition $a_3 + 1 = 8$ or a multiple of 8 has just been excluded we may show that $a_4 + 1 \neq 8$ or a multiple of 8. *Therefore as a preliminary result we have that* $a_4 + 1$, *the exponent of* p_4, *is either* 2 *or* 4.*

The plan of proof in the following paragraphs of this section will be to show that $2^{a_1+1} - 1$ contains both the factors p_3 and p_4, and that therefore $a_2 = 1$, and then to examine that case. From equation (5) we have

$$(8) \qquad \frac{2^{a_1+1}}{2^{a_1+1}-1} \cdot \frac{3^{a_2+1}}{3^{a_2+1}-1} = \frac{p_3^{a_3} + \cdots + p_3 + 1}{p_3^{a_3}} \cdot \frac{p_4^{a_4} + \cdots + p_4 + 1}{p_4^{a_4}}.$$

Hence

$$(9) \qquad \left(1 + \frac{1}{2^{a_1+1}-1}\right)\left(1 + \frac{1}{3^{a_2+1}-1}\right) \geqq \left(1 + \frac{1}{p_3}\right)\left(1 + \frac{1}{p_4}\right).$$

The only possible prime factors of $2^{a_1+1} - 1$ are 3, p_3, p_4. If $2^{a_1+1} - 1$ contains the factor 3 it is readily seen, by aid of the congruence $2^{a_1+1} \equiv 1 \pmod 3$, that $a_1 + 1$ is a multiple of 2; say $= 2y$. If $y > 2$, characteristic factors of $2^y - 1$ and $2^{2y} - 1$ will necessarily be p_3 and p_4. If $y = 2$, the number

* The reader will observe that practically the same reasoning applies with a similar conclusion to any case in which neither p_1 nor p_2 is greater than 5.

$2^{a_1+1}-1$ contains the prime 5, and therefore $p_3=5$; and then from inequality (9) it is easy to show that $p_4>3^{a_2+1}-1$. Then $3^{a_2+1}-1$ contains no prime factors except 2 and 5, and therefore $a_2=1$ or 3. Then from (9) we should have

$$3>\frac{3\cdot 5}{2^3}\cdot\frac{2^3\cdot 5}{3^3}\cdot\frac{2\cdot 3}{5},$$

which is not true; and hence $y\neq 2$. If $y=1$, equation (5) may be reduced to

$$(10)\qquad 2=\frac{3^{a_2+1}-1}{2\cdot 3^{a_2}}\cdot\frac{p_3^{a_3+1}-1}{p_3^{a_3}(p_3-1)}\cdot\frac{p_4^{a_4+1}-1}{p_4^{a_4}(p_4-1)},$$

a condition which can be satisfied only when $3^{a_2}p_3^{a_3}p_4^{a_4}$ is a multiply perfect odd number of three different primes. But it is known that no such numbers exist,* and hence *when* 3 *is a divisor of* $2^{a_1+1}-1$ *so also are* p_3 *and* p_4.

In the next case suppose that $2^{a_1+1}-1$ is not divisible by 3. Then a_1+1 is odd. We shall now show that $2^{a_1+1}-1$ is not a power of either p_3 or p_4. If

$$2^{a_1+1}-1=p_i^a,\text{ we have }2^{a_1+1}=p_i^a+1=\frac{p_i^{2a}-1}{p_i^a-1}.$$

Any characteristic factor of the numerator (when it has one) is by the lemma of the form $2as+1$, and hence >2. The denominator does not contain this factor. Also $p_i^{2a}-1$ has always a characteristic factor except when $a=1$ and p_i is of the form 2^v-1. Hence in every case $a=1$, and therefore $2^{a_1+1}-1=p_i$; that is, if $2^{a_1+1}-1$ is the power of a prime it is the first power. Then if $2^{a_1+1}-1$ is not divisible by 3 and contains only one of the primes p_3, p_4, say p_i, $2^{a_1+1}-1=p_i$. Denote the other by p_k. Then by inequality (9), $p_k>3^{a_2+1}-1$; hence $3^{a_2+1}-1$ is not divisible by p_k. By a second application of inequality (9) it is easy to show that if p_i divides $3^{a_2+1}-1$, then $p_i=2^{a_1+1}-1<p_k$; and therefore $p_i=p_3$ and $p_k=p_4$.

Proceeding to consider this case we should have $3^{a_2+1}-1=xp_3^{\mu}$ where μ and x represent integers and $\mu<a_3$ and $x\geqq 2$. Then from (8) we have by a simple transformation

$$(11)\qquad \left(1+\frac{1}{p_3}\right)\left(1+\frac{1}{xp_3^{\mu}}\right)\geqq\left(1+\frac{1}{p_3}+\cdots+\frac{1}{p_3^{a_3}}\right),$$

* Carmichael, *American Mathematical Monthly*, vol. 13, p. 35; 1906.

from which it follows that

$$(12) \qquad 1 + \frac{1}{p_3} + \frac{1}{xp_3^{\mu}} + \frac{1}{xp_3^{\mu+1}} \geqq 1 + \frac{1}{p_3} + \cdots + \frac{1}{p_3^{a_3}} .$$

But since both $2^{a_1+1} - 1$ and $3^{a_2+1} - 1$ contain p_3, $a_3 \geqq 2$. Moreover $\mu < a_3$ and also $x \geqq 2$. Hence inequality (12) is not satisfied except when $\mu = 1$. In this case we have $3^{a_2+1} - 1 = 2\,(2^{a_1+1} - 1)$; or $3^{a_2+1} = 2^{a_1+2} - 1$, an equation which cannot be satisfied since $a_1 \neq 0$. It follows therefore that $p_i = 2^{a_1+1} - 1$ is not a divisor of $3^{a_2+1} - 1$. But p_k does not divide $3^{a_2+1} - 1$ and therefore this number has no prime factor except 2 when $2^{a_1+1} - 1$ is a prime. Therefore $a_2 + 1 = 2$. Since $p_k > 3^{a_2+1} - 1$ it is equal to or greater than 11. Hence from (1) we may readily deduce, since $a_3 = 1$,

$$(13) \qquad 3 < \frac{2^{a_1+1} - 1}{2^{a_1}} \cdot \frac{4}{3} \cdot \frac{11}{10} \cdot \frac{2^{a_1+1} - 1}{2^{a_1+1} - 2} = \frac{44}{15} \, \frac{2^{2a_1+2} - 2^{a_1+2} + 1}{2^{2a_1+2} - 2^{a_1+2}} .$$

From this it follows readily that

$$2^{2a_1+2} - 2^{a_1+2} < 44,$$

and since $2^{a_1+1} - 1$ must be prime, it is easy to show that $2^{a_1+1} - 1 = 3$. This will lead again to an equation of the form of (10), and the case of this paragraph is therefore excluded. Hence *there exist no multiply perfect numbers of the type* $2^{a_1} 3^{a_2} p_3^{a_3} p_4^{a_4}$ *for which* $2^{a_1+1} - 1$ *is the power of a prime, so that when* $2^{a_1+1} - 1$ *is not divisible by* 3 *it contains both* p_3 *and* p_4. Combining this result with that of a preceding paragraph of this section we have that *for a multiply perfect number of the type* $2^{a_1} 3^{a_2} p_3^{a_3} p_4^{a_4}$ *and multiplicity* 3, $2^{a_1+1} - 1$ *must contain both* p_3 *and* p_4.

Now from inequality (9) it is easy to show by a method used in the preceding paragraphs that p_3 and p_4 each is greater than $3^{a_2+1} - 1$ and hence that $3^{a_2+1} - 1$ does not contain either p_3 or p_4. Then $3^{a_2+1} - 1$ must in every case be a power of 2, and hence by the lemma, $a_2 + 1 = 2$.

We proceed to investigate the values of $2^{a_1+1} - 1$ which contain two prime factors different from 3 and which might occur in equation (8). From (9) it follows readily that both p_3 and p_4 are $\geqq 11$, both being greater than $3^{a_2+1} - 1 = 8$, from (1) we have in the present case

$$(14) \qquad 3 < 2 \cdot \frac{4}{3} \cdot \frac{p_3}{p_3 - 1} \cdot \frac{p_4}{p_4 - 1} ,$$

from which it follows that $p_4 \leqq 43$. By inspection for smaller values of α_1 and with the help of the lemma for larger values, we find that there is only one value of $2^{\alpha_1+1}-1$, namely $2^{10}-1 = 3 \cdot 11 \cdot 31$, which has two prime factors other than 3 and lying between 11 and 43, the limits 11 and 43 being included. For this value we have

$$(15)\quad 3 = \frac{3 \cdot 11 \cdot 31}{2^9} \cdot \frac{2^2}{3} \cdot \frac{11^{\alpha_3} + \cdots + 11 + 1}{11^{\alpha_3}} \cdot \frac{31^{\alpha_4} + \cdots + 31 + 1}{31^{\alpha_4}}.$$

We have above shown that $\alpha_4 = 1$ or 3; and therefore by inspection of equation (15) it is easy to see that $\alpha_4 = 1$. Likewise $\alpha_3 = 1$ and the equation is satisfied; and we therefore have the result that *the number* $2^9 \cdot 3 \cdot 11 \cdot 31$ *is the only multiply perfect number of the type* $2^{\alpha_1} 3^{\alpha_2} p_3^{\alpha_3} p_4^{\alpha_4}$ *and of multiplicity* 3.

3. Non-existence of even multiply perfect numbers of the type $2^{\alpha_1} 5^{\alpha_2} p_3^{\alpha_3} p_4^{\alpha_4}$ and of multiplicity 3. In this case by the same reasoning as at the beginning of the preceding section,* it may be shown that $\alpha_4 = 1$ or 3. With $m = 3$, $p_1 = 2$, $p_2 = 5$, it is readily seen from (2) that $p_3 = 7$ and $p_4 \leqq 31$. Then $2^{\alpha_1+1}-1$ can contain no prime > 31; also it has only one prime other than 3, 5, 7. Hence by a liberal use of the lemma it turns out that $\alpha_1 + 1 = 2, 3, 4, 5, 6, 8$, or 12. The values 6 and 12 are excluded by their introducing too many 3's into the numerator of the right-hand member of (1). From (1) we may write

$$3 < \frac{2^{\alpha_1+1}-1}{2^{\alpha_1}} \cdot \frac{5}{4} \cdot \frac{7}{6} \cdot \frac{p_4}{p_4 - 1}.$$

By aid of this inequality the values $\alpha_1 + 1 = 2, 3$ are readily excluded. There are left $\alpha_1 + 1 = 4, 5, 8$, which we shall examine in turn.

For $\alpha_1 + 1 = 4$, we have from (1)

$$(16)\quad 3 = \frac{3 \cdot 5}{2^3} \cdot \frac{5^{\alpha_2} + \cdots + 5 + 1}{5^{\alpha_2}} \cdot \frac{7^{\alpha_3} + \cdots + 7 + 1}{7^{\alpha_3}} \cdot \frac{p_4^{\alpha_4} + \cdots + p_4 + 1}{p_4^{\alpha_4}}.$$

Now no prime > 31 can be introduced. If $5^{\alpha_2} + \cdots + 5 + 1$ contains 7 it contains 3, which by (16) is inadmissible. Hence $p_4^{\alpha_4} + \cdots + 1$ must con-

* See footnote page 152.

tain 7^{a_3}. If $p_4 = 11, 17, 19, 23, 29$, or 31, in each case we may readily show that an inadmissible prime will occur when $p_4^{a_4} + \cdots + p_4 + 1$ contains 7. Hence $p_4 = 13$, and since $a_4 = 1$ or 3, it is easy to show that $a_4 = 1$. Then $a_3 = 1$, and hence also $a_2 = 1$, and 3 occurs the second time in the numerator of (16); and this is inadmissible.

For $a_1 + 1 = 5$, equation (1) becomes

$$(17)\qquad 3 = \frac{31}{2^4}\cdot\frac{5^{a_2}+\cdots+5+1}{5^{a_2}}\cdot\frac{7^{a_3}+\cdots+7+1}{7^{a_3}}\cdot\frac{31^{a_4}+\cdots+31+1}{31^{a_4}}.$$

Since $a_4 = 1$ or 3 it is evident that the numerator of (17) contains too high a power of 2. In a similar manner we may show that when $a_1 + 1 = 8$ the resulting numerator contains too high a power of 3, p_4 in this case being equal to 17. This completes the discussion of numbers of the type $2^{a_1} 5^{a_2} p_3^{a_3} p_4^{a_4}$, no multiply perfect number being found. Combining the last result with those of section 2 we have the theorem that

The only even multiply perfect number of (only) four different primes and of multiplicity 3 is the number $2^9 \cdot 3 \cdot 11 \cdot 31$ previously found.

4. Non-existence of odd multiply perfect numbers of only four different primes. Let the number be $N = p_1^{a_1} p_2^{a_2} p_3^{a_3} p_4^{a_4}$ where $p_1 < p_2 < p_3 < p_4$ are different primes. Then by inequality (2) we see at once that for any odd multiply perfect number the multiplicity must be 2, while $p_1 = 3$ and $p_2 =$ either 5 or 7.

First let $p_2 = 7$. Then by (2), $p_3 = 11$ or 13.

If $p_3 = 13$, $p_4 = 17$ or 19, as may be readily shown by (2). For this case write (1) as follows:

$$2 = \frac{3^{a_1}+\cdots+3+1}{3^{a_1}}\cdot\frac{7^{a_2}+\cdots+7+1}{7^{a_2}}\cdot\frac{13^{a_3}+\cdots+13+1}{13^{a_3}}\cdot\frac{p_4^{a_4}+\cdots+p_4+1}{p_4^{a_4}}.$$

By a liberal use of the lemma it may be shown from the last equation that $p_4 \neq 19$; and that if $p_4 = 17$, $a_4 = 1$. This introduces the only admissible 2 into the numerator above; moreover it is easy to show that if any numerator contains 17 the now inadmissible 2 is introduced, and hence this case yields no numbers of the type under consideration.

Let $p_3 = 11$. Then by (2) $p_4 \leqq 23$; and therefore $p_4 = 13, 17, 19$, or 23. By using the lemma we exclude the values 17, 19, 23; and if $p_4 = 13$

we may readily show that $a_4 = 1$. This introduces into the numerator of (1) the only admissible 2. The factor $11^{a_3} + \cdots + 11 + 1$ in (1) must contain 3, 7, or 13. But if it contains 3 or 13, the inadmissible 2 occurs; and if it contains 7, 19 is introduced making it impossible to satisfy (1). This case therefore yields no numbers of the type considered here. Hence we conclude that there are no multiply perfect numbers of the type $3^{a_1} 7^{a_2} p_3^{a_3} p_4^{a_4}$, $p_4 > p_3 > 7$.

We shall now take up the case when $p_2 = 5$. By the same reasoning as that which was used at the beginning of section 2, it follows that in this case also $a_4 = 1$ or 3. But if $a_4 = 3$, the numerator of the second member of (1) will contain too high a power of 2, and therefore $a_4 = 1$. Then from (1) we may write

$$(18)\quad 2 = \frac{3^{a_1} + \cdots + 3 + 1}{3^{a_1}} \cdot \frac{5^{a_2} + \cdots + 5 + 1}{5^{a_2}} \cdot \frac{p_3^{a_3} + \cdots + p_3 + 1}{p_3^{a_3}} \cdot \frac{p_4 + 1}{p_4}.$$

None of the quantities a_1, a_2, a_3 can be odd; for then the numerator of (18) would contain too many 2's. Now by the lemma the characteristic factors of $3^{a_1+1} - 1$, $5^{a_2+1} - 1$, $p_3^{a_3+1} - 1$ must each be > 5; for $a_1 + 1$, $a_2 + 1$, $a_3 + 1$ are all odd numbers $\gtreqq 3$. Then it is easy to see by inspection of equation (18) that

$$(19)\qquad p_3^{a_3} + \cdots + p_3 + 1 = p_4,$$

$$(20)\qquad (3^{a_1} + \cdots + 3 + 1)(5^{a_2} + \cdots + 5 + 1) = p_3^{\mu},$$

and then further that

$$(21)\qquad 2 \cdot 3^{a_1} \cdot 5^{a_2} p_3^{a_3 - \mu} = p_4 + 1.$$

From (20) it may be shown with little difficulty that

$$(22)\qquad 3^{a_1} 5^{a_2} \cdot \frac{3}{2} \cdot \frac{5}{4} > p_3^{\mu}; \quad \text{or} \quad 2 \cdot 3^{a_1} \cdot 5^{a_2} > \frac{16}{15} p_3^{\mu};$$

and then from (21) that

$$\frac{16}{15} p_3^{a_3} < p_4 + 1.$$

But from (19), $p_3^{a_3} \cdot \dfrac{p_3}{p_3 - 1} > p_4$, and hence $\dfrac{p_3}{p_3 - 1} > \dfrac{16}{15}$, from which it may be shown that $p_3 \gtreqq 13$. And now since p_3 is a characteristic factor of both $3^{a_1+1} - 1$ and $5^{a_2+1} - 1$, neither $a_1 + 1$ nor $a_2 + 1$ can be greater than 6; and

since each is odd and > 1, the only possible values are 3, 5. Moreover these values are to be so selected that $3^{a+1} - 1$ and $5^{a_2+1} - 1$ shall have the common factor $p_3 = 7, 11$, or 13, and neither shall have a factor $> p_3$. These conditions are not satisfied in any case. Combining this result with a previous one of this section we are able to announce that *there are no odd multiply perfect numbers of only four different primes.*

Collecting all the results of this paper and certain others referred to in the opening paragraphs, we have the theorem:

There exist but two multiply perfect numbers of (only) four different primes: these two are $2^5 \cdot 3^3 \cdot 5 \cdot 7$, *of multiplicity* 4; *and* $2^9 \cdot 3 \cdot 11 \cdot 31$, *of multiplicity* 3.

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ON A SYSTEM OF PARASTROIDS*

By R. P. Stephens

Introduction. If three points t_i be taken on a circle, say the unit circle, and if a fourth point τ of the same circle be projected at a fixed inclination k^2 on the three lines joining the points t_i, then these three projections lie on a line. As τ varies, this line envelops a three-cusped hypocycloid; by varying the angle of projection, Dr. Converse obtained a system of hypocycloids which he has discussed in detail.† In an earlier paper,‡ Prof. Steggall chose his points t_i so as to form a regular inscribed triangle, but he extended his investigation in this very special case to n points forming the vertices of a regular inscribed polygon of n sides. It is my intention to extend Dr. Converse's general treatment for three points to n points and to discuss at length the case where $n = 4$.§ For this purpose the algebraic work is greatly simplified by expressing the line equation of a curve in conjugate coordinates‖ and the point equation by means of mapping from the unit circle.

1. The Wallace Lines. The conjugate equation of a line through the points t_1 and t_2 of the unit circle is

$$(1) \qquad x + t_1 t_2 y = t_1 + t_2.$$

The reflexion of a point τ of the unit circle in this line is

$$x = t_1 + t_2 - \frac{t_1 t_2}{\tau}.$$

If now we represent the symmetric functions of t_i (where $i = 1, 2, 3$) by σ_1, σ_2, and σ_3, then this last equation becomes

$$(2) \qquad x = \sigma_1 - t - \frac{\sigma_3}{t\tau},$$

* Parts of this article appeared in the Johns Hopkins University Circular, January, 1905 (New Series, No. 1); other parts were presented to the American Mathematical Society, Dec. 27, 1906.

† H. A. Converse, Annals of Mathematics, ser. 2, vol. 5 (1904), pp. 105–139.

‡ Steggall, *Proceedings of the Edinburgh Mathematical Society*, vol. 14 (1895–6), p. 131.

§ This extension of Dr. Converse's work to four points was suggested by Professor Bromwich.

‖ F. Morley, *Trans. Amer. Math. Soc.*, vol. 1 (1900), p. 97, also vol. 4 (1904), pp. 1–12. H. A. Converse, loc. cit.

which for t equal to t_1, t_2, or t_3 gives the reflexion of τ in the line through the other two t's; hence, for varying t, this is the equation of some locus which passes through the three reflexions. The conjugate of (2) is

$$\sigma_3 y = \sigma_2 - \frac{\sigma_3}{t} - t\tau.$$

On eliminating t between (2) and its conjugate there results

$$(W_3) \qquad \tau x - \sigma_3 y = \sigma_1 \tau - \sigma_2,$$

the conjugate equation of a line, *i. e.*, the Wallace* line of the three points t_i as to τ. For convenience, I shall designate this line by W_3, the subscript referring to the number of t's.

If now four points t_i be used, there will be four lines W_3, obtained from the points t_i three at a time. The reflexion of τ in these four lines will be of the form

$$\tau x = \sigma_1 \tau - \sigma_2 + \frac{\sigma_3}{\tau}.$$

If this be made symmetrical for four t's, that is, if the substitution

$$\sigma_1 = \sigma_1' - t,$$
$$\sigma_2 = \sigma_2' - t\sigma_1' + t^2,$$
$$\sigma_3 = \sigma_4'/t,$$

be made, this gives

$$\tau x = (\sigma_1 - t)\,\tau - \sigma_2 + \sigma_1 t - t^2 + \sigma_4/t\tau,$$

where the accent has been dropped from σ_i and they now refer to four t's.

On eliminating t between this equation and its conjugate, there results

$$\tau^2 x + \sigma_4 y = \sigma_1 \tau^2 - \sigma_2 \tau + \sigma_3 - \frac{\tau}{t^2}\,(t^4 - \sigma_1 t^3 + \sigma_2 t^2 - \sigma_3 t + \sigma_4),$$

or, since $\qquad t^4 - \sigma_1 t^3 + \sigma_2 t^2 - \sigma_3 t + \sigma_4 = 0,$

$$(W_4) \qquad \tau^2 x + \sigma_4 y = \sigma_1 t^2 - \sigma_2 t + \sigma_3,$$

* Jno. S. MacKay, "The Wallace Line and the Wallace Point," *Proceedings of the Edinburgh Math. Soc.*, vol. 9 (1890–91). M. Cantor, *Geschichte der Math.*, vol. 3, p. 542.

the equation of the Wallace line for four points t_i as to τ. In a similar manner, we obtain, in general,

$$(W_n) \qquad \tau^{n-2}x + (-1)^n\sigma_n y = \sigma_1\tau^{n-2} - \sigma_2\tau^{n-3} + \sigma_3\tau^{n-4} \cdots + (-1)^n\sigma_{n-1},$$

the equation of the Wallace line* for n points t_i.

2. Some Curves of Class n, arising from the n Lines W_{n-1}. With these Wallace lines at hand it is now easy to write down the projection of a point τ on them at any fixed inclination.

Consider the equation of the line

$$(1) \qquad x + t_1t_2y = t_1 + t_2;$$

then

$$x + k^2t_1t_2y = \tau + \frac{k^2t_1t_2}{\tau}$$

is a line which meets (1) at a fixed inclination † and passes through the point τ. These two lines intersect at the point

$$(k^2-1)x = k^2\left(t_1 + t_2 - \frac{t_1t_2}{\tau}\right) - \tau,$$

which is therefore the *projection* of the point τ on the line (1) at a fixed inclination.

On symmetrizing this equation for three t's as on p. 160 we obtain

$$(k^2-1)x = k^2\left(\sigma_1 - t - \frac{\sigma_3}{t\tau}\right) - \tau,$$

or, eliminating t between this and its conjugate, we find

$$(C_3) \qquad \tau^3 - [k^2\sigma_1 - (k^2-1)x]\,\tau^2 + [k^2\sigma_2 + k^2(k^2-1)\sigma_3 y]\tau - \sigma_3k^4 = 0,$$

the equation of a line, namely, the line on which the three projections must lie; but, as τ varies, C_3 envelops the deltoid which formed the basis of Dr. Converse's article referred to above.

* This derivation of the Wallace lines is due to Professor Morley and was given by him in a course of lectures in 1903–4.

† If θ is the angle which the second line makes with the first, then $k^2 = e^{2i\theta}$; if the second makes the supplementary angle to θ with the first

$$k_1^2 = e^{2i(\pi-\theta)} = e^{-2i\theta} = \frac{1}{k^2}.$$

In a similar manner if we start with n fixed points on the circle and project a point τ of the same circle on the n lines W_{n-1} arising from the fixed points taken $n-1$ at a time, we obtain

$$(C_n) \qquad \tau^n - [k^2\sigma_1 - (k^2-1)x]\,\tau^{n-1} + k^2\sigma_2\tau^{n-2} + \cdots$$
$$- (-1)^n \{[k^2\sigma_{n-1} + k^2(k^2-1)\,\sigma_n y]\,\tau - k^4\sigma_n\} = 0,$$

the equation of a line.

Therefore the n projections of a point of a circle on the n Wallace lines, arising from n points of the circle taken $n-1$ at a time, lie on a line.

By the method explained later in the particular case where $n = 4$, it is found that *the curve enveloped by C_n, as τ varies, is a curve of class n and order $2n-2$, and has, in general, n cusps whose n cusp-tangents touch a curve of class $n-2$.*

3. The Equation of the Parastroid. Let us consider more particularly the equation C_4, which, if the axis of reals be so chosen that $\sigma_4 = 1$, may be written

$$(3) \qquad \tau^4 - [k^2\sigma_1 - (k^2-1)x]\,\tau^3 + k^2\sigma_2\tau^2 - [k^2\sigma_3 + k^2(k^2-1)y]\,\tau + k^4 = 0.$$

For a fixed value of τ, this is the equation of a line, but as τ varies the line envelops a curve which is of *fourth class*, since from any given point there are obviously four tangents. Its map equation, obtained by dividing by τ and then differentiating with respect to τ, is

$$(4) \qquad 2\left(x - \frac{k^2\sigma_1}{k^2-1}\right) = \frac{k^2}{k^2-1}\left[\left(\frac{k^2}{\tau^3} - \frac{3\tau}{k^2}\right) - \frac{\sigma_2}{\tau}\right],$$

in which k is a constant turn and τ is the variable turn. Since a line cuts it in six points, the curve is of the *sixth degree*. The curve is readily shown to be the parallel to an astroid, that is, it is a *parastroid*.*

From both (3) and (4) it is evident that the center of the curve is

$$x_0 = \frac{k^2\sigma_1}{k^2-1}.$$

If, however, k is allowed to vary, we shall obtain a system of parastroids the locus of whose centers is

$$x = \frac{k^2\sigma_1}{k^2-1},$$

*G. Loria, *Spezielle ebene Kurven*, p. 651.

a straight line perpendicular to the stroke σ_1 at its mid-point. Therefore, *the centers of all the parastroids of the system, obtained by varying k, lie on the right line perpendicular to the stroke σ_1 at it mid-point* (see figure 1).

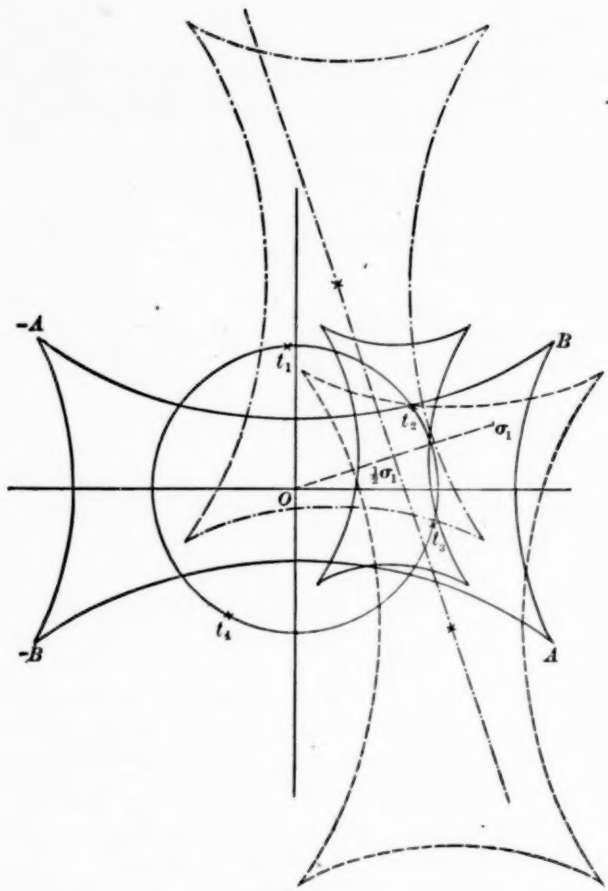

FIG. 1.

A similar theorem may be stated for the general case, that is, for the locus of the centers of the curves enveloped by the line C_n as k varies.

4. The Cusps of the Parastroid. The condition for cusps is obtained from (4) by equating $dx/d\tau$ to zero. It is

$$\tau^4 - \tfrac{1}{3}\,\sigma_2 k^2\tau^2 + k^4 = 0, \tag{5}$$

of which the roots are:

$$\tau_1 = ak, \qquad \tau_3 = -ak,$$
$$\tau_2 = k/a, \qquad \tau_4 = -k/a,$$

where $2a \equiv \sqrt{\frac{1}{3}\sigma_2 + 2} + \sqrt{\frac{1}{3}\sigma_2 - 2}$.

These four values of τ are turns (*i. e.*, restricted to the unit circle) for all values of σ_2 such that $\sigma_2 \gtreqless 6$, — a condition which, according to the formation of σ_2, is always satisfied. Putting these values of τ in (4), we find as the four cusps of that parastroid for which $k = k_1$:

$$x_1 = \frac{k_1A + k_1^2\sigma_1}{k_1^2 - 1}, \qquad x_3 = \frac{-k_1A + k_1^2\sigma_1}{k_1^2 - 1},$$

$$x_2 = \frac{k_1B + k_1^2\sigma_1}{k_1^2 - 1}, \qquad x_4 = \frac{-k_1B + k_1^2\sigma_1}{k_1^2 - 1},$$

where

$$2A \equiv \frac{1}{a^3} - 3a - \frac{\sigma_2}{a}$$

and B is the conjugate of A. The points $-A$, $-B$, A, and B are readily shown to be the cusps of the parastroid

$$(6) \qquad \tau^4 - x\tau^3 + \sigma_2\tau^2 - y\tau + 1 = 0,$$

where σ_2 has the value given above.

By the substitution of $1/k^2$ for k^2 in equation (3), we obtain the equation of the parastroid which results from the projection of τ at an angle, the supplement of the angle of projection used above. Making this substitution in the values of the cusps (6), we find the cusps of the new parastroid, x_1', x_2', x_3', and x_4', each of which when added to its correspondent in (6) gives σ_1. Therefore, *the two parastroids are symmetrical with respect to* $\sigma_1/2$.

If in (6) k_1 is allowed to vary, then x_1 traces out the curve

$$(8) \qquad x = \frac{kA + k^2\sigma_1}{k^2 - 1},$$

which is a hyperbola with center at $\sigma_1/2$. Comparing the locus of x_1 with that of x_3 we see that when k of the first, as it runs around the unit circle, becomes

$-k$, we have the third cusp; so these two cusps trace out the same curve. Similarly, the second and the fourth cusps trace out the same hyperbola

$$x = \frac{kB + k^2\sigma_1}{k^2 - 1}, \tag{9}$$

whose center is also at $\sigma_1/2$.

Therefore, *the four cusps of the parastroids of the system lie on two fixed hyperbolas which are concentric* (see figure 2).

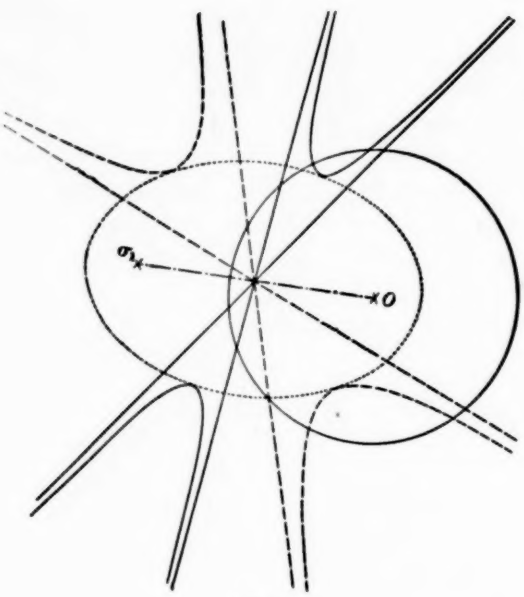

FIG. 2.

The four cusps of any parastroid of the system lie on the circle

$$[(k^2 - 1)x - \sigma_1 k^2][(k^2 - 1)y + \sigma_3] + k^2 AB = 0; \tag{10}$$

but for a varying k, this represents a system of circles whose envelope is the ellipse

$$(\sigma_3 x + \sigma_1 y - \sigma_1\sigma_3 + AB)^2 - 4AB\,xy = 0, \tag{11}$$

with center at $\sigma_1/2$, with foci at origin and σ_1, and with major axis $\sqrt{AB}$.

We can therefore say that *the cusp-circles of the system of parastroids envelop an ellipse whose center is σ_1, whose foci are the origin and σ_1 and whose major axis is equal to the absolute value of A* (see figure 2).

5. The Size, Shape, and Motion of the Parastroid. For the sake of clearness it will be well to think of the system of parastroids as if it were but a *single parastroid moving* with its center on a fixed line and changing its size according to some fixed law.

The astroid to which the parastroid (3) is parallel is

$$\tau^4 - [k^2\sigma_1 - (k^2 - 1)x]\,\tau^3 - [k^2\sigma_3 + k^2(k^2 - 1)y]\tau + k^4 = 0.$$

For any given value of τ, this equation and equation (3) represent two parallel straight lines, whose distance apart is one-half the absolute value of the difference between their constant terms, *i. e.*, $\frac{1}{2}\,|k^2\sigma_2\tau^2|$ or $\sigma_2/2$, a constant independent of both τ and k. Therefore, since the shape of a parastroid is entirely dependent on the distance of its generating line from the corresponding line of the astroid to which it is parallel, *the shape of every parastroid of our system is fixed when σ_2 is given;* or, putting it differently, *as the parastroid moves, it retains its shape* (see figure 1).

New let us notice the stroke between two adjacent cusps, say $x_1 - x_2$. From (6) we find

$$x_1 - x_2 = \frac{k(A - B)}{k^2 - 1}.$$

The clinant* of this stroke is

$$\frac{x_1 - x_2}{y_1 - y_2} = \frac{\dfrac{k(A - B)}{k^2 - 1}}{\dfrac{-k(B - A)}{k^2 - 1}} = 1.$$

In the same way,

$$\frac{x_1 - x_4}{y_1 - y_4} = -1.$$

From these, it is seen that the first stroke $x_1 - x_2$ is always parallel to the axis of reals, and the second $x_1 - x_4$ is always perpendicular to it; that is, *the parastroid moves parallel to itself with its four cusps forming a rectangle, a side of which is always parallel to the axis of reals* (see figure 1).

* Franklin, "On some applications of circular coordinates," *Amer. Jour. of Math.*, vol. 12(1890), p. 162.

The area of a parastroid of constant shape varies as the square of the absolute value of a diagonal, say $x_1 - x_3$. Then from (6), we obtain

$$|x_1 - x_3|^2 = -\frac{4k^2}{(k^2-1)^2} AB,$$

as the square of a diagonal. From the expression for the center of a parastroid, we obtain as the square of the distance of this center from the origin

$$|x_0|^2 = -\frac{k^2}{(k^2-1)^2}\sigma_1\sigma_3.$$

On comparing this with the square of a diagonal, there results a constant,

$$\frac{|x_1 - x_3|^2}{|x_0|^2} = \frac{4AB}{\sigma_1\sigma_3}.$$

Hence, *the area of the moving parastroid varies directly as the square of the distance of its center from the origin.*

6. The Cusp Loci — the two Hyperbolas. The equations of these two hyperbolas have already been given. They are

$$x = \frac{kA + k^2\sigma_1}{k^2-1}, \tag{8}$$

$$x = \frac{kB + k^2\sigma_1}{k^2-1}. \tag{9}$$

Since both have the same form, it will be necessary to consider but one of them. The first (8) may be written

$$x = \frac{kA}{k^2-1} + \frac{k^2\sigma_1}{k^2-1},$$

which shows the curve as the resultant of two simpler curves; namely,

1) $$x_1 = \frac{kA}{k^2-1},$$

the equation of a line (counted twice) which passes through the origin, perpendicular to the stroke A, but with the part from $+ iA/2$ to $- iA/2$ omitted; and

2) $$x_2 = \frac{k^2\sigma_1}{k^2-1},$$

the equation of a line perpendicular to the stroke σ_1 at its mid-point. These two equations can be easily combined, for, on giving k some particular value k_1, we have

$$|x_1| = \sqrt{\frac{-k_1^2}{(k_1^2-1)^2}AB},$$

$$|x_2| = \sqrt{\frac{-k_1^2}{(k_1^2-1)^2}\sigma_1\sigma_3},$$

whose ratio is evidently the ratio of the absolute values of the strokes A and σ_1. Thus, corresponding points on the two loci can be marked and added so as to form the points of the hyperbola.*

The asymptotes of (8) are readily found. The equation of any line through the point $\sigma_1/2$ is

$$2x + 2ty = \sigma_1 + t\sigma_3.$$

On substituting (8) and its conjugate in this equation, we obtain those values of k which give the points of intersection of the line and the hyperbola. They are the roots of the equation.

$$2(kA + \sigma_1 k^2 - tkB - t\sigma_3) = (k^2 - 1)(\sigma_1 + t\sigma_3).$$

But it is seen from (8) that, when $k = \pm 1$, x is infinite; hence these values of k give the clinants t of those lines through the center $\sigma_1/2$ which cut the hyperbola at infinity. In this way, we find

$$t_1 = \frac{\sigma_1 - A}{\sigma_3 - B}, \qquad t_2 = \frac{\sigma_1 + A}{\sigma_3 + B}.$$

Therefore the two asymptotes of the hyperbola (8) are

(12) $$2\left(x + \frac{\sigma_1 \pm A}{\sigma_3 \pm B}\, y\right) = \sigma_1 + \frac{\sigma_1 \pm A}{\sigma_3 \pm B}\,\sigma_3,$$

where the upper signs give one and the lower signs the other asymptote.

In a similar manner, the asymptotes of (9) are found to be

(13) $$2\left[x + \frac{\sigma_1 \pm B}{\sigma_3 \pm A}\, y\right] = \sigma_1 + \frac{\sigma_1 \pm B}{\sigma_3 \pm A}\,\sigma_3.$$

* This gives a very simple construction of the hyperbola whose map equation has the form (8).

If lines be drawn from the point σ_1 to the two cusps A and $-A$ of (7), these strokes are $\sigma_1 - A$ and $\sigma_1 + A$, whose clinants are respectively

$$\frac{\sigma_1 - A}{\sigma_3 - B} \quad \text{and} \quad \frac{\sigma_1 + A}{\sigma_3 + B}.$$

On comparing these with the clinants of the asymptotes (12), it is seen that they differ only in sign. Therefore, *the asymptotes of the hyperbola* (8) *are perpendicular to the strokes from* σ_1 *to the two opposite cusps* A *and* $-A$ *of the parastroid* (7). (See figure 3).

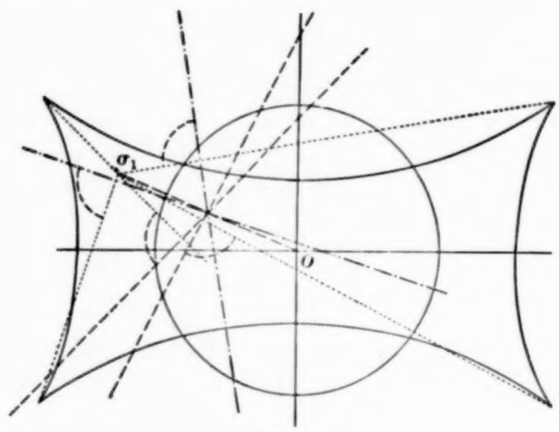

FIG. 3.

In a similar manner, it is shown that *the asymptotes of* (9) *are perpendicular to the strokes from* σ_1 *to the other two cusps* B *and* $-B$ *of* (7).*

The hyperbola (8) will be rectangular when its asymptotes are perpendicular, that is, when from (12)

$$\frac{\sigma_1 + A}{\sigma_1 + B} = -\frac{\sigma_1 - A}{\sigma_3 - B}, \quad \text{or} \quad \sigma_1\sigma_3 = AB.$$

Therefore, *if the given points* t_i *on the unit circle are such that* σ_1 *is at one of the cusps of the parastroid* (7), *the hyperbola* (8) *will be rectangular.*

* A similar theorem is true for the case of 3 fixed t's, discussed by Dr. Converse, and may be stated thus: *The three asymptotes of the cusp-locus of the deltoids of the system* C_3 *are perpendicular to the strokes from* σ_1 *to the three cusps of the deltoid*

$$t^3 - xt^2 + \sigma_3 yt - \sigma_3 = 0.$$

Since the same condition holds for (9), both hyperbolas are rectangular at the same time.

The two following theorems are easily established:

The four common tangents of the two hyperbolas form a rectangle, a side of which is parallel to the axis of reals (see figure 4).

The ellipse, enveloped by the cusp-circles of the system of parastroids,

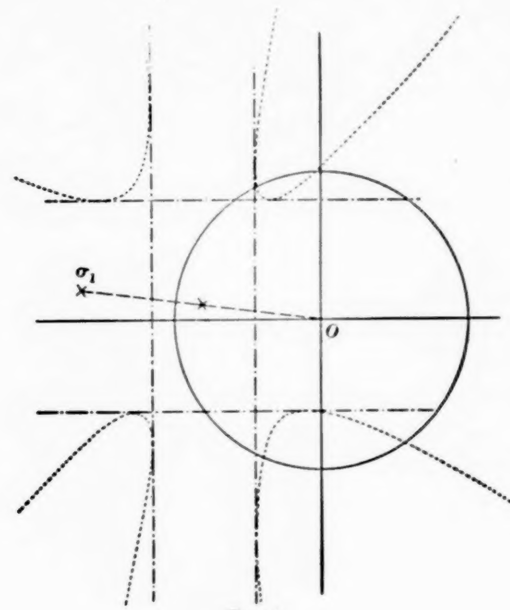

FIG. 4.

touches each of the two hyperbolas, the locus of cusps, in two points (see figure 2).

7. Special Shapes of the Parastroid. From what was said in §5, it is evident that the shape of any parastroid of the system (3) is the same as that of the parastroid (7), so here for the sake of simplicity we need only discuss the shapes of the latter, that is, the shape of

$$(7) \qquad \tau^4 - x\tau^3 + \sigma_2\tau^2 - y\tau + 1 = 0.$$

This will evidently be an astroid when $\sigma_2 = 0$, so the only condition to be imposed on the four given points t_i of the unit circle is that $\sigma_2 = 0$.

The parastroid will be tangent to itself when it passes through its own center, which in this case is the origin. Therefore, in the map equation of (7), on putting x equal to zero, we find

$$\sigma_2 = -3\tau^2 + \frac{1}{\tau^2}.$$

But since σ_2 must always be real when $\sigma_4 = 1$, it must be equal to its conjugate, so

$$\sigma_2 = -3\tau^2 + \frac{1}{\tau^2} = -\frac{3}{\tau^2} + \tau^2,$$

or

$$\tau = \pm 1.$$

On putting this value for τ above, there results

$$\sigma_2 = \pm 2,$$

which is the necessary and sufficient condition that the parastroid be tangent to itself.

The only remaining shape of a parastroid of special interest is the one whose cusps have coincided by twos, forming an ellipse-shaped curve.* Obviously the condition here is (see p. 163)

$$A = B.$$

On substituting the values of A and B in terms of σ_2, this reduces to the form

$$\sigma_2 = \pm 6,$$

which is the condition on the t's that the curve may have the desired shape.

The fixed points on the unit circle may be readily chosen in order to satisfy any one of these three conditions.

WESLEYAN UNIVERSITY,
MIDDLETOWN, CONN.,
FEBRUARY, 1906.

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* For properties of this curve see Wolstenholme's *Mathematical Problems*, p. 303.

A PECULIAR EXAMPLE IN MINIMA OF SURFACES

By E. R. Hedrick

The theory of maxima and minima of a surface abounds in paradoxes of an instructive nature. One such, first given by Peano,* is now commonly cited: †

The surface

$$z = (y - x^2)(y - 2x^2) \tag{1}$$

has neither a maximum nor a minimum at the origin; yet every plane section of this surface by a plane through the z axis has an isolated proper minimum at the origin.

This is explained readily by a figure, which is essentially the same as that suggested below. It is evident that *cylinders* may be drawn which cut this surface in curves which do not have minima at the origin; for example, the circular cylinder $x^2 + (y - 1/3)^2 = 1/9$ gives a curve of intersection which actually has a maximum at the origin.

An example of even more curious behavior is the following:

The surface

$$(2) \quad \begin{aligned} z &= (y - e^{-\frac{1}{x^2}})(y - 2e^{-\frac{1}{x^2}}) && \text{when } x \neq 0 \\ z &= y^2 && \text{when } x = 0 \end{aligned}$$

is continuous, and z possesses finite partial derivatives of every order at every point. The surface has neither a maximum nor a minimum at the origin, yet every section by a right cylinder whose trace upon the xy plane

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* A. Genocchi and G. Peano, *Calcolo differenziale e principii di calcolo integrale*, 1884, p. XXIX; German translation by Bohlmann and Schepp, 1899, p. 332. Cf. p. III of the German translation.

† See, e. g., Goursat, *Course in Mathematical Analysis* (English trans.), vol. 1, p. 124. A similar property with reference to the distance from a point to a surface will be given by the writer in a paper to appear in the *Bulletin of the American Mathematical Society*.

*is an analytic curve through the origin has an isolated proper minimum at the origin.**

The function $f(x) = e^{-\frac{1}{x^2}}$ (defined $= 0$ for $x = 0$), was first used by Cauchy, and is now a standard example of a curious function of a single variable;† $f(x)$, together with all its derivatives, is defined and continuous for all values of x; it is positive for $x \neq 0$; but all its derivatives vanish at $x = 0$, as does $f(x)$ itself. It follows that $f(x)$ is not expansible in Taylor's infinite series at $x = 0$.

Returning to the example (2) itself, it is seen that of the two factors of z, the first is positive for $y > e^{-\frac{1}{x^2}}$, the second for $y < e^{-\frac{1}{x^2}}$; hence if the two curves

$$C_1: \quad y = e^{-\frac{1}{x^2}},$$

$$C_2: \quad y = 2e^{-\frac{1}{x^2}},$$

be drawn, z is negative between C_1 and C_2, is zero on C_1 and on C_2, and otherwise is positive.

It is now easy to see that any analytic curve K through $(0, 0)$ in the xy plane cannot penetrate the region between C_1 and C_2 inside of a sufficiently small circle about $(0, 0)$.

First, suppose that K is not tangent to the x axis at $(0,0)$; then evidently K has only the one point $(0, 0)$ in common with C_1 and C_2 and otherwise lies wholly outside the region between the two, near the origin. Secondly, along the x axis itself, z is evidently positive except at the origin.

Finally, if K is tangent to the x axis at the origin, but does not coincide with it, then K has contact of some finite order with the x axis at $(0, 0)$; hence the ordinate of K is negative or else it is greater than the corresponding ordinate of $y = 2e^{-\frac{1}{x^2}}$ for every value of x different from zero, near the origin, since C_2 itself has contact of infinitely high order with the x axis. The formal proof of this statement is easy, using Taylor's theorem with a remainder. In fact if K has an equation of the form $y = \phi(x)$ near $x = 0$, and if $\phi(x)$ pos-

* I shall understand by an analytic curve a curve which at each point may be represented by an equation of the form $y = \phi(x)$ [or $x = \psi(y)$], where $\phi(x)$ [or $\psi(y)$] is expansible by Taylor's infinite series at that point. This is easily shown to be equivalent to a definition in parameter form; compare e. g., a paper by Ames, *American Journal of Math.*, vol. 27 (1905), p. 343.

† See, e. g., Goursat, l. c., p. 106.

sesses a derivative of any order which does not vanish at (0, 0), then z is positive along K except at (0, 0), even when $\phi(x)$ is non-analytic.

The original statement concerning the surface (2) is thus proved. It may be noted that each of the partial derivatives of z (of all orders) vanishes at (0, 0) except $\partial^2 z/\partial y^2$, which is identically equal to 2. It is evident, therefore, that none of the usual tests by means of partial derivatives would lead to any conclusion in this instance; for

$$B^2 - AC = z_{xy}^2 - z_{xx} z_{yy} = 0$$

at (0, 0), and all derivatives of order higher than the second vanish. It is also evident that z is not expansible in Taylor's double infinite series, for the sum of the series obtained by attempted expansion is y^2, which is equal to z only for $x = 0$, $y = 0$.

Models of this surface and of that of Peano are readily constructed in plaster; these at once allay the natural incredulity which at first attaches to such paradoxes.

COLUMBIA, MO., APRIL, 1907.

ON MAXIMUM AND MINIMUM VALUES OF THE MODULUS OF A POLYNOMIAL*

By D. N. Lehmer

In this note we consider the values of z of given modulus which give maximum or minimum values to the modulus of a given polynomial (rational integral function). Let the function be

$$f(z) = \sum_{i=0}^{n} p_i z^{n-i},$$

where each p_i is real. Put $z = \rho\ (\cos\theta + i\sin\theta)$. The corresponding value of the function is $x + i\,y$, where

$$\begin{aligned} x &= \sum_{i=0}^{n} p_i\,\rho^{n-i}\cos\,(n-i)\theta, \\ y &= \sum_{i=0}^{n} p_i\,\rho^{n-i}\sin\,(n-i)\theta. \end{aligned} \tag{1}$$

If the modulus of the point $x + i\,y$ is to be a minimum or a maximum for a given ρ, we must have

$$x\,\frac{dx}{d\theta} + y\,\frac{dy}{d\theta} = 0, \tag{2}$$

or, after a few trigonometric reductions,

$$\sum_{\substack{i=0\\ j=0}}^{n} p_i\,p_j\,\rho^{2n-i-j}\,(i-j)\sin\,(i-j)\theta = 0. \tag{3}$$

If the point z moves along the circumference of a circle of radius ρ with center at the origin, the point (x, y) will describe a curve parametrically given by equations (1). Equation (3) solved for θ gives the amplitudes of those points on this circle which correspond to the feet of normals from the origin on the curve described by (x, y).

* Read before the San Francisco Section of the American Mathematical Society, under a slightly different title, February 24, 1906.

Suppose now ρ to vary. On each circle ρ there will be a certain number of points of the above description giving a locus of which (3) is the polar equation. This locus transforms into the locus of the feet of the normals from the origin upon the family of curves which corresponds to the system of circles ρ. From the equation (2) it is clear that this curve passes through the root points of $f(z)$.

THEOREM. *The lines joining the root points of $f(z)$ to the origin are tangent at those points to the curve whose polar equation is* (3), *or else form part of that curve.*

To prove this theorem we note that

$$\frac{\partial x}{\partial \rho}\frac{\partial x}{\partial \theta} + \frac{\partial y}{\partial \rho}\frac{\partial y}{\partial \theta} \equiv 0, \tag{4}$$

since, as is easily verified,

$$\frac{\partial x}{\partial \rho}\frac{\partial x}{\partial \theta} = -\frac{\partial y}{\partial \rho}\frac{\partial y}{\partial \theta} = -\sum p_i p_j (n-i)(n-j)\rho^{2n-i-j-1} \sin(2n-i-j)\theta.$$

Now from

$$x\frac{\partial x}{\partial \theta} + y\frac{\partial y}{\partial \theta} = 0$$

we get, differentiating with respect to ρ,

$$\frac{\partial x}{\partial \rho}\frac{\partial x}{\partial \theta} + \frac{\partial y}{\partial \rho}\frac{\partial y}{\partial \theta} + x\frac{\partial^2 x}{\partial \theta \partial \rho} + y\frac{\partial^2 y}{\partial \theta \partial \rho} = 0,$$

whence by (4) we have

$$x\frac{\partial^2 x}{\partial \rho \partial \theta} + y\frac{\partial^2 y}{\partial \rho \partial \theta} = 0,$$

which vanishes for $x = y = 0$. If therefore in equation (3) a value of θ be taken which will make the vector pass through a root point of $f(z)$, the equation will have a double root ρ equal to the modulus of that root. The vector ρ is therefore either a tangent to the curve or else forms part of the curve.

It is not difficult to prove the more general theorem that if p is a k-fold root of $f(z)$, the vector from the origin to p touches the curve (3) in $k+1$ coincident points or else as before forms part of the curve.

BERKELEY, CALIFORNIA,
FEBRUARY, 1907.

ON THE MINIMUM SURFACE OF REVOLUTION IN THE CASE OF ONE VARIABLE END POINT

By Mary Emily Sinclair

Introduction. In the problem of minimizing the integral

$$J = \int_{x_0}^{x_1} y\sqrt{1 + y'^2}\,dx,$$

that is, of finding a minimum surface of revolution when the end point $P_1(x_1, y_1)$ of the generating curve is fixed and the end point $P_0(x_0, y_0)$ is permitted to vary along a curve

$$D: \quad y = g(x),$$

the solution of Euler's differential equation gives the set of extremals *

$$(1) \qquad E: \quad y = a \operatorname{Ch} \frac{x - b}{a}.$$

We determine b as a function of a,

$$b = x_1 - a \operatorname{Ch}^{-1} \frac{y'}{a},$$

by the condition that the curve (1) passes through P_1, thus obtaining the one-parameter set of extremals through P_1,

$$(2) \qquad y = a \operatorname{Ch} \frac{x - b(a)}{a}.$$

The condition of transversality † gives, for our problem, at the point P_0,

$$(3) \qquad 1 + y'g' = 0,$$

that is, the extremal must be perpendicular to the curve D.

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* We use Laisant's notation for hyperbolic functions.

† Kneser, *Lehrbuch der Variationsrechnung*, §10. Bolza, *Lectures on the Calculus of Variations*, §23, (33).

Assuming that (3) is fulfilled for at least one extremal E_0 of the set E, we denote by P_0 the point common to D and E_0. It is then further necessary that the focal point of the curve D shall not lie between P_0 and P_1. Any of

FIG. 1.

the general formulae * give for the determination of the focal point, $P(\bar{x}, \bar{y})$, the following equation: †

$$(4) \qquad \frac{\phi(\bar{u})-\phi(u_0)}{\phi(\bar{u})+\mathrm{Sh}\,u_0\,\mathrm{Ch}\,u_0+u_0} = -\frac{\rho}{\epsilon a\,\mathrm{Ch}^2 u_0\,\mathrm{Sh}\,u_0},$$

where

$$\bar{u} = \frac{\bar{x}-b}{a}, \qquad u_0 = \frac{x_0-b}{a},$$

$$\phi(u) = \mathrm{Coth}\,u - u,$$

$$\epsilon = \frac{\mathrm{Sh}\,u_0}{|\mathrm{Sh}\,u_0|},$$

and ρ is the radius of curvature of D at P_1 with the usual agreement as to sign.‡ It is evident that $\epsilon = \pm 1$ as $u_0 \gtrless 0$.

In this paper we propose (1) to give a detailed *discussion of the transcendental equation* (4) for the determination of the focal point, (2) to obtain a *simple geometric construction for the focal point* which shall include the Lindelöf§ construction as a special case when $\rho = 0$, and (3) to give a *physical interpretation of the focal point* as defining for the case $\rho = \infty$ the limit of stability of a liquid film.‖

* Kneser, loc. cit., §23. Bolza, loc. cit., §23. Bliss, *Transactions of the Amer. Math. Soc.*, vol. 3 (1902), p. 132.

† Given in slightly different form by Kneser, loc. cit., p. 85, equation (65).

‡ Compare, for instance, Scheffers, *Anwendungen der Differential- und Integralrechnung auf Geometrie*, vol. 1, p. 37.

§ See, for instance, Bolza, loc. cit., p. 64.

‖ See Plateau, *Statiques des liquides*, in particular §90, 1 and §§111, 225–227.

1. Discussion of the transcendental equation (4). In order to discuss equation (4), we shall first study the function

$$\phi(u) = \operatorname{Coth} u - u. \tag{5}$$

Since $\phi'(u) = -\operatorname{Coth}^2 u < 0$, $\phi(u)$ is a decreasing function of u. Also

$$\phi''(u) = \frac{2\operatorname{Ch} u}{\operatorname{Sh}^3 u} \gtrless 0$$

according as $u \gtrless 0$. Further $\phi(u)$ becomes infinite if and only if $u = 0$ or ∞. Hence, $\phi(u)$ is a continuous, decreasing function which takes every value k once and but once for $u > 0$ and once and but once for $u < 0$. The curve is as shown in figure 2.

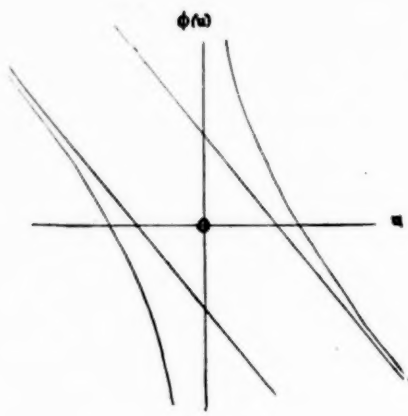

Fig. 2.

We now consider ρ as a function of $\bar{u}$, writing (4) as follows:

$$\rho = -\epsilon a \operatorname{Ch}^2 u_0 \operatorname{Sh} u_0 \frac{\phi(\bar{u}) - \phi(u_0)}{\phi(\bar{u}) - \phi(v_0)}, \tag{6}$$

where v_0 and v_0' are the positive and negative quantities respectively, defined by the equations

$$\phi(v_0) = \phi(v_0') = -(u_0 + \operatorname{Ch} u_0 \operatorname{Sh} u_0). \tag{7}$$

Then $$\frac{d\rho}{d\bar{u}} = -\epsilon a \operatorname{Ch}^5 u_0 \phi'(\bar{u}) \frac{1}{[\phi(\bar{u}) - \phi(v_0)]^2}.$$

We assume throughout $a > 0$, and consider two cases, $u_0 < 0$ and $u_0 > 0$.

Case 1. $u_0 < 0$. Then $\epsilon = -1$, and therefore $d\rho/d\bar{u} < 0$, whence ρ is everywhere a decreasing function of $\bar{u}$. Further, ρ is a linear function of $\phi(\bar{u})$, and therefore takes every value twice as $\bar{u}$ varies from $-\infty$ to $+\infty$. We shall denote by u_0' the value of $\bar{u}$ at P_0', the point conjugate to P_0, u_0' being determined by the equation,

$$\phi(u_0') - \phi(u_0) = 0.*$$

From (6) the values taken by $\bar{u}$ when $\rho = \infty$ are v_0 and v_0', where $v_0 < u_0$,

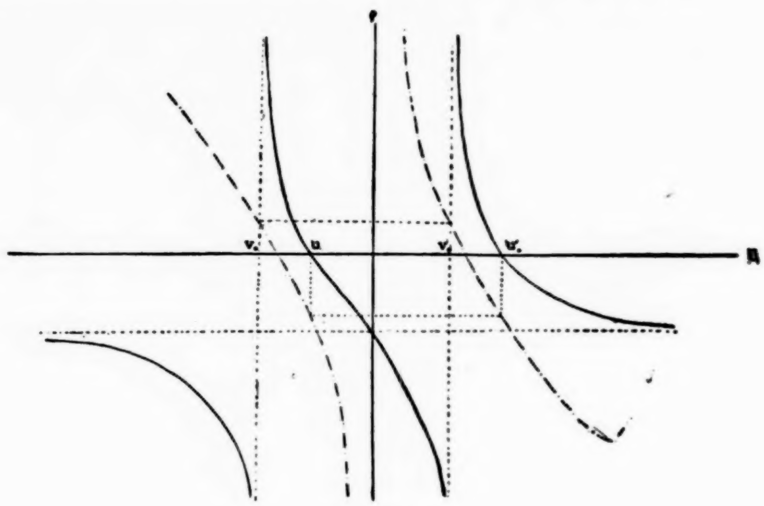

FIG. 3.

$v_0' < u_0'$, since from (5) and (7) $\phi(v_0) > \phi(u_0)$ and $\phi(v_0') > \phi(u_0')$. We obtain, then, the following table and the curve in figure 3:

$\bar{u}$	$-\infty$	v_0	u_0	0	v_0'	u_0'	$+\infty$
$\phi(\bar{u})$	$+\infty$	$\phi(v_0)$	$\phi(u_0)$	$\mp\infty$	$\phi(v_0)$	$\phi(u_0)$	$-\infty$
ρ	M	$\mp\infty$	0	M	$\mp\infty$	0	M,

where $M = a\,\mathrm{Ch}^2 u_0\,\mathrm{Sh}\,u_0$.

As $\bar{u}$ increases from u_0 to u_0', ρ decreases continuously from 0 to $-\infty$ and again from $+\infty$ to 0. Conversely, for every value of ρ between $-\infty$ and $+\infty$ there exists one and only one corresponding value of $\bar{u}$ between u_0 and u_0'. Remembering that $x = au + b$, we conclude that as ρ decreases from 0 to

* See, for instance, Bolza, loc. cit., p. 64, (32).

$-\infty$ and again from $+\infty$ to 0, $\bar{x}$ increases continuously from x_0 to x_0' and therefore $\bar{P}_0$ describes the catenary in positive sense from P_0 to P_0'. Hence, every value of ρ determines one and only one point of E_0 focal to the curve D and lying on the interval $P_0\,P_0'$.*

Case 2. $u_0 > 0$. Here $\operatorname{Sh} u_0 > 0$ and $\epsilon = +1$, and therefore $d\rho/d\bar{u} > 0$ and ρ is everywhere an increasing function. The root u_0' of the equation $\phi(u) - \phi(u_0) = 0$ is negative, and therefore less than u_0. Hence in this case there exists no point conjugate† to P_0 with an abscissa greater than that of P_0. The values taken by $\bar{u}$ when $\rho = \infty$ are v_0 and v_0' where $v_0 > u_0$ and $v_0' > u_0'$, since from (5) and (7) again $\phi(v_0) < \phi(u_0)$ and $\phi(v_0') < \phi(u_0')$. We obtain then the following table:

$\bar{u}$	$-\infty$	u_0'	v_0'	0	u_0	v_0	$+\infty$
$\phi(\bar{u})$	$+\infty$	$\phi(u_0)$	$\phi(v_0)$	$\mp\infty$	$\phi(u_0)$	$\phi(v_0)$	$-\infty$
ρ	$-M$	0	$\pm\infty$	$-M$	0	$\pm\infty$	$-M$,

where $M = a\operatorname{Ch}^2 u_0 \operatorname{Sh} u_0$.

We study the interval $u_0 \cdots \infty$. As $\bar{u}$ increases from u_0 to ∞, ρ increases from 0 to $+\infty$ and again from $-\infty$ to $-a\operatorname{Ch}^2 u_0 \operatorname{Sh} u_0$. Conversely, to

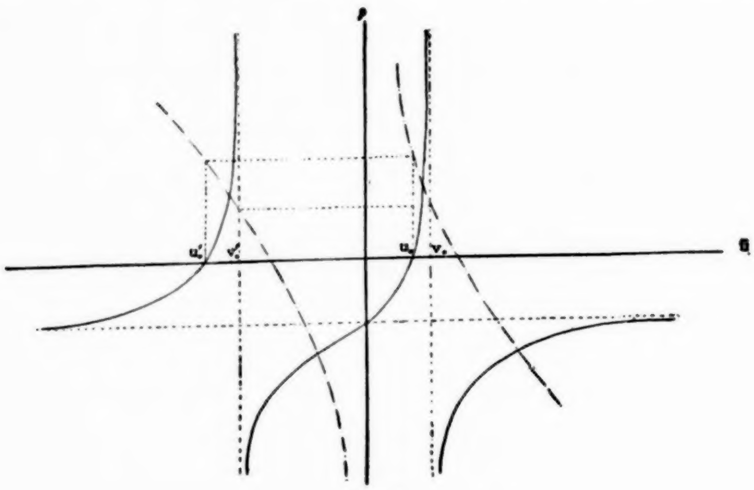

FIG. 4.

* In accordance with a general theorem due to Bliss, *Trans. Amer. Math. Soc.*, vol. 3 (1902), p. 139.

† See Kneser, loc. cit., p. 89. Bolza, loc. cit., §14.

every value of ρ between $-a \operatorname{Ch}^2 u_0 \operatorname{Sh} u_0$ and 0 there exists no corresponding $\bar{u}$ on the interval, but for all other values of $\bar{u}$ between $-\infty$ and $+\infty$, there exists one and only one corresponding value of $\bar{u}$ on the interval (see figure 4). As ρ increases from 0 to $+\infty$ and from $-\infty$ to $-a \operatorname{Ch}^2 u_0 \operatorname{Sh} u_0$, $\bar{x}$ increases continuously from x_0 to ∞, and therefore $\bar{P}_0$ describes the catenary in positive sense from P_0 to ∞. When ρ lies between $-a \operatorname{Ch}^2 u_0 \operatorname{Sh} u_0$ and v, there exists no focal point $\bar{P}_0$ with an abscissa greater than that of P_0.

2. A Geometrical construction for the focal point. We proceed to obtain a geometric construction for the focal point $\bar{P}_0$, treating separately the two cases discussed above.

Case 1. $u_0 < 0$. From (6) we have:

$$\frac{a[\phi(\bar{u}) - \phi(v_0)]}{a[\phi(u_0) - \phi(v_0)]} = -\frac{a \operatorname{Ch}^2 u_0 \operatorname{Sh} u_0}{\rho - a \operatorname{Ch}^2 u_0 \operatorname{Sh} u_0}. \tag{8}$$

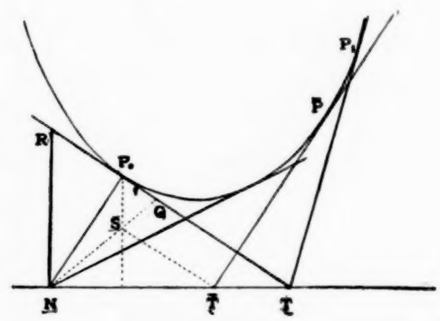

FIG. 5.

Let the tangent and normal at P_0 cut the x axis at T and N respectively, and let R be the point on the tangent whose abscissa is that of N.

Then $$NT = -a\,[\phi(u_0) - \phi(v_0)]$$

and $$|P_0R| = -a \operatorname{Ch}^2 u_0 \operatorname{Sh} u_0.$$

Further if $\bar{T}$ is the point of intersection of the tangent at the focal point $\bar{P}_0$ with the x-axis, then

$$N\bar{T} = -a\,[\phi(\bar{u}) - \phi(v_0)].$$

Equation (8) then becomes

$$\frac{N\bar{T}}{NT} = \frac{|P_0R|}{\rho + |P_0R|}, \tag{9}$$

and $\bar{T}$ may now be located for any assigned value of ρ by the following construction :—

On P_0T lay off $P_0Q = \rho$. Draw NQ and let the ordinate at P intersect NQ in S. Draw $S\bar{T}$ parallel to RT. Of the two tangents which can be drawn from $\bar{T}$ to the catenary, one will meet the curve at a point $\bar{P}_0$ between P_0 and P_0', and $\bar{P}_0$ is the focal point.

Three special cases are of interest.

(*a*) $\rho = 0$. Here P_0 coincides with S, $\bar{T}$ with T, and $\bar{P}_0$ with P_0'. The construction reduces to the Lindelöf * construction for the conjugate of P_0, the equation being the well-known one,

$$\phi(\bar{u}) - \phi(u_0) = 0.$$

(*b*) $\rho = \infty$. Here $\bar{T}$ coincides with N, the equation being

$$\phi(\bar{u}) - \phi(v_0) = 0,$$

and we obtain the following simple rule:

When $\rho = \infty$, $\bar{P}_0$ is the point of tangency of that tangent from N which meets the curve between P_0 and P_0'.

(*c*) $\rho = a\,\mathrm{Ch}^2 u_0\,\mathrm{Sh}\, u_0$. Here Q coincides with R, and $\bar{T}$ is at an infinite distance on the x-axis. The focal point is therefore the vertex of the catenary.

Now let ρ decrease continuously from 0 to $-\infty$ and again from $+\infty$ to 0. $\bar{T}$ then describes the x-axis in positive sense, starting from T when $\rho = 0$ and passing through the infinite point of the x-axis when $\rho = a\,\mathrm{Ch}^2 u_0\,\mathrm{Sh}\, u_0$, and through N when $\rho = \mp\infty$, returning to T for the value $\rho = 0$. At the same time $\bar{P}_0$ describes P_0P_0' in positive sense.†

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* Lindelöf, *Mathematische Annalen*, vol. 2 (1870), p. 160. Bolza, loc. cit., p. 64.

† I have generalized these results for the case

$$J = \int_{x_0}^{x_1} y^k \sqrt{1 + y'^2}\, dx.$$

Euler's equation gives here

$$y' = \sqrt{\left(\frac{y}{a}\right)^{2k} - 1}.$$

Denoting the solution of this equation by $y = a\,\phi(u)$ where $u = (x - b)/a$, we obtain

Case 2. $u_0 > 0$. In this case, (6) becomes

$$\frac{a[\phi(\bar{u}) - \phi(v_0)]}{a[\phi(u_0) - \phi(v_0)]} = \frac{a\,\mathrm{Ch}^2 u_0\,\mathrm{Sh}\,u_0}{\rho + a\,\mathrm{Ch}^2 u_0\,\mathrm{Sh}\,u_0}. \tag{8'}$$

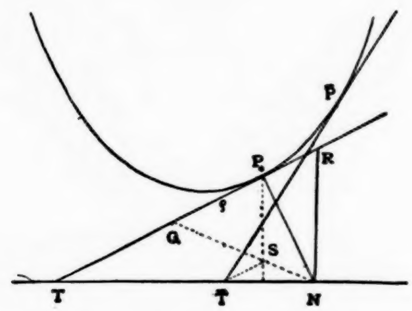

FIG. 6.

Let the tangent and normal at P_0 cut the x-axis at T and N respectively, and let R be the point on the tangent whose abscissa is that of N.

Then $$NT = -a[\phi(u_0) - \phi(v_0)],$$

and $$|P_0 R| = a\,\mathrm{Ch}^2 u_0\,\mathrm{Sh}\,u_0.$$

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for the determination of the focal point the equation

$$\frac{-a\,\phi_0\phi_0'(1 + \phi_0'^2)^{1/2}}{k\rho} = \frac{\dfrac{\bar{\phi}}{\bar{\phi}'} - \bar{u} + u_0 + \phi_0\phi_0'}{\dfrac{\bar{\phi}}{\bar{\phi}'} - \bar{u} - \dfrac{\phi_0}{\phi_0'} + u_0},$$

where ϕ_0 and u_0 are used to refer to the initial point and $\bar{\phi}$ and $\bar{u}$ for the focal point. Interpreted geometrically, this equation is:

$$-\frac{|P_0R|}{k\rho} = \frac{N\bar{T}}{N\bar{T} - NT},$$

whence $$\frac{N\bar{T}}{NT} = \frac{|P_0R|}{\rho k + |P_0R|}.$$

If therefore, $k\rho$ is used in place of ρ in the construction given in the text, $\bar{T}$ is determined in this more general case, and the point of contact of the tangent from $\bar{T}$ to the extremal is again the focal point.

And, if $\overline{T}$ is again the point of intersection of the tangent at the focal point with the x-axis, then $N\overline{T} = -a[\phi(\bar{u}) - \phi(v_0)]$. Equation (8′) becomes

$$(8) \qquad \frac{N\overline{T}}{NT} = \frac{|P_0R|}{\rho + |P_0R|},$$

and $\overline{T}$ may now be located for any assigned value of ρ by the same construction as in case 1. *When $u_0 > 0$ the construction of the point T is similar to that given above, but of the two tangents which can be drawn from $\overline{T}$ to the catenary, that one must be chosen which meets the curve at a point $\overline{P}_0$ whose abscissa is greater than that of P_0. $\overline{P}_0$ is then the focal point.* Such a point always exists for all values of ρ except values $-a\,\mathrm{Ch}^2 u_0\,\mathrm{Sh}\,u_0 < \rho < 0$, as we showed in §1.

3. Physical illustration for the case $\rho = \infty$. The theory of minimum surfaces may be beautifully illustrated by experiments with liquid films. An extended discussion of such experiments is to be found in Plateau's treatise. The following simple experiment* illustrates the theory of the focal point. A glass funnel A is held in upright position and a smaller one B is inverted within it so that its rim is in contact with the inner surface of A. Soap solution is then applied at this rim, and the funnel B is withdrawn vertically, care being taken to maintain an axis of symmetry. A cathetometer is used for the necessary measurements. A catenoid film is formed, extending from the lateral surface of the funnel A to the rim of B. Its lower opening creeps up the inner surface of the funnel A, equilibrium being found when the angle between the film and the funnel A is 90°. The film is stationary and

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* See Plateau's experiments, loc. cit. Two wire rings are placed in contact, moistened with soap solution, and drawn apart. A catenoid film then extends from one to the other, and possesses perfect stability up to a certain limit, the length of axis being then about two thirds the diameter of the rings. Plateau's experimental results are compared below with those of Lindelöf which are given by the theory of the conjugate point [see *Math. Ann.*, vol. 2 (1870) p. 160].

	diameter of ring.	length of axis.	diameter of neck.
Theoretical value	71.49	47.38	39.47
Experimental value	71.49	46.85	39.60

A second experiment of Plateau's determines the limit of stability of a catenoid film formed by withdrawing vertically one ring from the surface of a soap solution. In this case the limit of stability is given by the theory of the focal point, when the curve D is a straight line parallel to the y-axis.

perfectly stable when any fixed height is maintained, so long as this height is within a certain limit. At the limit of stability, which occurs when the point P in figure 7 is the focal point of the generator D of the funnel A, the film gradually separates at the neck into two convex films which recede respectively upon the two funnels. Beyond this limit, the catenoid is not the surface of least area.

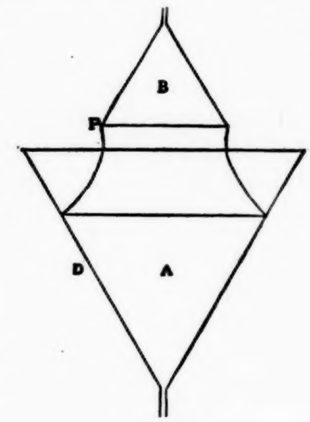

FIG. 7.

We shall give the mathematical solution of the problem and then compare it with experimental results. Since all catenaries are similar figures, we obtain the values when $a = 1$, and then adapt our values to the constants of the apparatus. As x-axis we use the axis of symmetry of the apparatus and as y-axis a perpendicular through the vertex of A. Let a be the semi vertical angle of A. Let P_0 be a point common to the catenary E and the line D, whose equations are as follows:

$$(10) \qquad D: \quad \begin{cases} x = r \cos a, \\ y = r \sin a; \end{cases}$$

$$(11) \qquad E: \quad \begin{cases} x = u + b, \\ y = \mathrm{Ch}\, u. \end{cases}$$

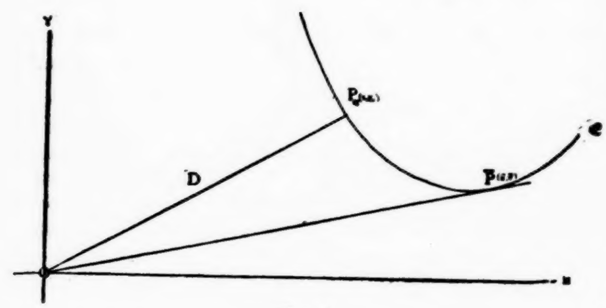

FIG. 8.

The point $\bar{P}_0$ is then determined by a tangent through the origin. We seek to determine x_0, y_0, b, $\bar{x}_0$, $\bar{y}_0$.

For E,

$$y' = \mathrm{Sh}\, u = \epsilon \sqrt{y^2 - 1},$$

where $\epsilon = \dfrac{|\mathrm{Sh}\, u|}{\mathrm{Sh}\, u}$ and the sign $\sqrt{\ }$ is used always for the positive square root.

Since, by the condition of transversality, E and C are perpendicular at P_0, we have

$$\cos a - \sin a \sqrt{y_0^2 - 1} = 0. \quad \text{Hence}$$

$$(12) \qquad y_0 = \csc a, \qquad r_0 = \csc^2 a, \qquad \bar{x}_0 = \cos a \csc^2 a.$$

Therefore
$$\csc a = \mathrm{Ch}\left(\frac{\cos a}{\sin^2 a} - b\right),$$
or

$$b = \frac{\cos a}{\sin^2 a} - \mathrm{Ch}^{-1}(\csc a). \tag{13}$$

Since $\overline{P}_0$ is determined by the tangent from the origin, we have

$$\mathrm{Coth}\,\overline{u}_0 = \overline{x}_0 = \overline{u}_0 + b. \tag{14}$$

If we can solve this transcendental equation we may determine $\overline{y}_0$ from $\overline{y}_0 = \mathrm{Ch}\,\overline{u}_0$. Equation (14) may be written, using the notation of §1, $\phi(\overline{u}) = b$, which we have shown admits of a unique positive solution $\overline{u}_0$.

Let us now obtain the values of the constants when $a = 30°$, the actual angle of the experiment.

From (12) we obtain $y_0 = 2$, $x_0 = 2\sqrt{3} = 3.464$, $u_0 = \mathrm{Ch}^{-1}2 = -1.317$, $b = 2\sqrt{3} - \mathrm{Ch}^{-1}2 = 4.781$, since u_0 is negative.

We now obtain $\overline{u}_0$ from the equation $\phi(\overline{u}_0) = 4.781$ by approximation.

u	e^{2u}	Coth u	$\phi(u)$
.200	1.492	5.070	4.870
.203	1.5004	4.9970	4.791
.204	1.5038	4.9690	4.765
.205	1.5069	4.9450	4.740
.210	1.522	4.830	4.620

Hence,
$$\overline{u} = .2034,$$
$$\overline{x} = 4.984,$$
$$\overline{y} = 1.021,$$
$$\overline{x} - x_0 = 1.520.$$

In our apparatus, $\overline{y} = 2.4$, the radius of the opening of the small funnel. Then we have:

a	a	y_0	x_0	u_0	b	$\overline{u}$	$\overline{x}$	$\overline{y}$	$\overline{x} - x_0$
30°	1	2	3.464	−1.317	4.781	.2034	4.984	1.021	1.520
30°	2.35	4.7	8.14	−3.095	11.235	.4780	11.712	2.4	3.572

the second row being found from the first by multiplying throughout by $\dfrac{2.4}{1.021}$.

The experimental results, in centimeters, are compared with the theoretical values in the following table:

	$2a$	$2y_0$	$\bar{x} - x_0$	$\bar{x}$
Th'l Value	4.70	9.40	3.572	11.712
Exp. I	4.7	9.45	3.65	11.78
II	4.7	9.45	3.70	11.83
III	4.7	9.45	3.67	11.80
IV	4.65	9.45	3.70	11.83
V	4.6	9.5	3.65	11.78
VI	4.7	9.5	3.70	11.83
Average	4.68	9.48	3.68	11.81
Error	0.4%	0.9%	3%	0.9%

Large experimental errors were to be expected from the lack of symmetry in the apparatus used.

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ON THE POLYNOMIAL CONVERGENTS OF A POWER SERIES *

By M. B. Porter

If $\sum_0^\infty a_n x^n$ be a power series having a circle of convergence whose radius is r and if $P_n(x) = \sum_0^n a_m x^m$ be its polynomial convergent of degree n, Abel has shown that for any given point x', $|x'| > r$, $|P_n(x')|$ will become definitely infinite if n becomes infinite over a suitable set of integers $n_1, n_2, \cdots$ We shall denote such a set by (n_i). Various questions present themselves in this connection. If $\lim_{n_i=\infty} |P_{n_i}(x')| = \infty$, does this same property hold for all the points of a region in which x' lies and if so what are the shapes of such regions? Moreover since $\sum_0^\infty a_n x^n$ converges uniformly in $|x| < r - \epsilon$ to an analytic function $F(x)$, it follows that $P_n(x)$ will have, when n is large enough, just as many zeroes in $|x| < r - \epsilon$ as $F(x)$ has, i. e., a limited number. Thus the question of the distribution of the remaining zeroes of $P_n(x)$ presents itself. Finally it is possible that $\lim P_n(x)$ exist throughout a region lying outside or containing the circle of convergence inside of it when n becomes infinite over the set (n_j). In the last case, when the regions overlap we can get an analytic extension of our power series merely by inserting parentheses.

While the general solution of the foregoing questions would doubtless be a very difficult matter, the following theorem applicable to all power series throws some light on the character of the results we may expect.

Theorem. If $\sum_0^\infty a_n x^n$ *have a circle of convergence of radius* $r (0 \leqq r \leqq M)$, *then it will always be possible to find a set of integers* (n_j), *such that* $\lim_{n_j=\infty} P_{n_j}(x)$ *becomes definitely infinite throughout any given region lying outside the circle of convergence, save perhaps at a finite number of points of the region.*

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* Presented at the meeting of the American Mathematical Society at Columbia, Mo., December 1, 1906.

If s denote any point outside the circle of convergence, and we consider the numbers $|a_0|$, $|a_1 s|$, $|a_2 s^2|$, $\cdots$ $|a_n s^n|$, $\cdots$, since some of these numbers increase indefinitely, it will be possible to reach one greater than all that preceed it; if we go on still farther in the set, we shall reach a second greater than all that preceed it, and so on indefinitely. Let us denote the numbers thus obtained by $|a_{n_1} s^{n_1}|$, $|a_{n_2} s^{n_2}|$, $\cdots$ $|a_{n_i} s^{n_i}|$, $\cdots$.

Consider the polynomials

$$[1] \qquad P_{n_i}(x) \equiv a_{n_i} x^{n_i} \left[\frac{a_0}{a_{n_i}} x^{-n_i} + \frac{a_1}{a_{n_i}} x^{-n_i+1} + \cdots + \frac{a_{n_i-1}}{a_{n_i}} x^{-1} + 1 \right]$$

$$\equiv a_{n_i} x^{n_i} \bar{P}_{n_i}(x) \qquad (i = 1, 2, 3, \cdots).$$

Throughout the region $R(|x| > |s|)$,

$$|\bar{P}_{n_i}(x)| < \frac{1}{1 - \left|\frac{s}{x}\right|}$$

and is consequently limited in the region $R'(|x| > |s| + \epsilon)$. Thus by a theorem of Arzelà's* we can pick out of the set (n_i) at least one sequence (n_j) such that the polynomials $\bar{P}_{n_j}(x)$ converge uniformly to a function $\bar{P}(x)$ analytic in R', as n_j becomes infinite. Evidently this function $\bar{P}(x)$ does not vanish identically in R since $\bar{P}_{n_j}(\infty) = 1$.

Thus by a well known theorem, each polynomial $\bar{P}_{n_j}$ will have only a limited number of zeroes (say m') in R' (when n_j is large enough) and these zeroes will condense on the m' zeroes of $\bar{P}(x)$ which lie in R'. From [1] it is clear that these m' zeroes will lie in the region $|s| < |x| \leqq 2|s|$.† Denote by m the number of zeroes of $\sum_0^\infty a_n x^n$ in $|x| < r - \epsilon$; then $P_{n_j}(x)$ will have exactly m zeroes in this region if n_j is large enough. Consider a set of points $[s_i]$ where $|s_{i+1}| < |s_i| < |s|$ and $\lim_{i=\infty} s_i = r$. To each s_{i+1} will correspond a set of polynomials which will be a *sub*set of the set corresponding to s_i and at the same time a subset of $[P_{n_j}]$. Thus we see that

* For a proof of this theorem see a paper by Arzelà, Annals of Mathematics, ser. 2, vol. 5 (1903), p. 53, or a paper by the author, vol. 6, p. 190, of the same journal.

† If $\lim_{n=\infty} \frac{a_{n+1}}{a_n}$ exist, it is easy to see from elementary considerations that $P_n(x)$ will have no roots in the region $|x| > r + \epsilon$ for a sufficiently large value of n.

From the set $[P_{n_j}]$ *we can always extract a set which will ultimately have all its zeroes, save* $m + m'$, *in the region* $r - \epsilon < |x| < |s_i|$. *Moreover there must be points* on the circle of convergence which are limiting points of the zeroes of these polynomials.* This does not exclude the possibility of there being other limiting points of the complete set $[P_{n_j}]$ in $r - \epsilon < |x| < |s_i|$ *not* on the circle of convergence.

We can go further and show that some of the polynomials $P_{n_j}(x)$ will take on values less than any number small at pleasure in the neighborhood of any point of the circle of convergence. Consider the set of functions $\phi_{n_j} = \dfrac{1}{P_{n_j}}$ in the region $|x| > r$. If any region surrounding a point on the rim of the circle of convergence existed throughout which $|\phi_{n_j}(x)| < M$ (fixed) for all values of the sequence n_j, then since for the part of this region in which $|x| > r + \epsilon$

$$\lim_{n_j=\infty} \phi_{n_j}(x) = 0,$$

it would follow that

$$\lim_{n_j=\infty} \phi_{n_j} = 0$$

inside the circle $|x| < r$ (by Arzelà's or Stieltje's theorem†), which is a contradiction.

As has already been remarked, it may happen that a sequence $(\bar{n}_i)$ can be found such that $\lim_{\bar{n}_i=\infty} P_{\bar{n}_i}(x)$ exists throughout regions outside the circle of convergence or throughout regions in which the circle of convergence lies.

To construct such series, write

$$[A] \qquad \sum_0^\infty c_{n_i} z^{n_i} \quad \text{(radius of convergence } r\text{)}.$$

Set in this series

$$z = Kx(1 + x);$$

* In case $r = 0$ then the point $x = 0$ will be such a limiting point; in the case $\sum_0^\infty x^n$ every point on the rim of the circle of convergence is a limiting point.

† For a statement of this theorem see a paper by Osgood, *Functions defined by Infinite Series*, ANNALS OF MATHEMATICS, ser. 2, vol. 3 (1901), p. 33.

then the circle of convergence $|z| = r$ will go over into the Cassinian $r = |Kx(1 + x)|$ inside of which

$$[A'] \qquad \phi(x) = \sum_0^\infty c_{n_i} K^{n_i} x^{n_i} (1 + x)^{n_i}$$

will converge absolutely. Any singularity of $[A]$ will go over into two singularities of $[A']$. Let us now suppose that $n_{i+1} > 2n_i$; then if the binomials in $[A']$ are multiplied out, it can be written as a power series in x merely by removing parentheses. The resulting power series will converge in a circle about the origin reaching up to the nearest singularity of $\phi(x)$. Evidently in the case before us this circle must lie wholly inside the Cassinian and hence it can only reach up to the nearest point on the Cassinian, which must consequently be a singularity of $\phi(x)$. Similarly if we make the transformation

$$[B] \qquad z = Ke^{\theta i} x(1 + x),$$

the same argument shows that *every* point on the circle of convergence $|z| = r$ goes over into a singular point of $\phi(x)$, so that the circle of convergence of $[A]$ is a natural boundary.*

The series $\phi(x)$ regarded as a power series in x is such that by the insertion of suitable parentheses it can be made to converge throughout the domain of existence of the function it defines. Moreover if the Cassinian consists of two ovals, it will converge in both ovals and define different analytic functions, in the sense that neither can be regarded as the analytic continuation of the other.

It may be noticed that the transformation $[B]$ used in constructing $\phi(x)$ reduces the investigation of singularities on the rim of the circle of convergence to a question concerning double series, so that to show that $re^{\theta i}$ is a singularity of $[A]$ it would be sufficient to prove that $[A']$ arranged as a power series in x converges for any positive real value for which it converged before it was so arranged.

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* This also follows from a theorem of Hadamard's; *La Série de Taylor*, p. 36.

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